International Journal of Theoretical Physics, Vol. 12, No. 4 (1975), pp. 225–240

Topics on Space-Time Topology (III)

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Received: 3 *July* 1974

Abstract

The paper is divided into two parts. The first part is a detailed review of the topological background to the theory of spinor structures on space-times, including proofs of most of the main results. The second part is a contination of the analysis in the second paper of this series of the cobordism theory of compact, closed, orientable space-time manifolds. A new cobordism relation is defined: linear spin cobordism and the linear spin cobordism of space-time manifolds calculated.

1. *Introduction*

This article is intended mainly as a detailed review of the topological background of the theory of spinor structures on space-time manifolds. I also report on some very recent work of Koschorke (1974) on the bordism of tangent line bundles which provides the answer to a question I posed (Whiston, 1974) concerning the cobordism of Lorentzian structures on compact space-times. These results have relevance to the geometry of space-times because of Steenrod's famous theorem on the correspondence between Lorentzian structures and tangent line bundles on manifolds. It turns out that any two Lorentzian structures on a given compact, closed, orientable four manifold are cobordant, moreover two Lorentzian structures on different closed, compact manifolds are cobordant iff the two manifolds are cobordant in the oriented sense. Therefore the bordism classification of Lorentzian structures is essentially trivial. In general, there are an infinite number of homotopy classes of tangent line bundles on a four-manifold (of Euler number zero if it is compact). By introducing a more refined definition: concordance of tangent line bundles (asking for a homotopy through tangent line bundles) Koschorke showed that there can be but a finite number of concordance classes, the number depending upon the $\mathbb{Z}_2$-dimension of the $\mathbb{Z}_2$-linear space $H^1(X, \mathbb{Z}_2)$. Below I define a new cobordism relation: linear spin cobordism between closed, compact spin-manifolds and obtain the following generalisation of results in (Whiston, 1974). Two compact, closed, spinor space-time manifolds X_0^{+,s_1} and X_1^{+,s_2} with $\text{sig}(X_0^+) = \text{sig}(X_1^+)$ are cobordant through a spin bordism of their tangent bundles: there is a compact five-dimensional spin-manifold $Z^{+,s}$ with oriented boundary $X_0^+ U X_1^-$ and a tangent four-plane bundle z^1 on Z with a spin-structure on its principal $SO(4)$ bundle which restricts to the tangent bundles of X_0^+ and X_1^- inducing the preassigned spin-structures (arising from the restriction of the spinor-structures to Spin (3).

The main review is divided into six sections. In the first two sections I present some background material on the Lorentz groups and their associated Clifford algebras and spinor groups, including an analysis of the pseudo-projective spaces and a few lemmas on the topology of the Lorentz and spinor groups. In the next two sections I discuss the theory of Lorentzian structures and spinor structures on four-manifolds in terms of homotopy theory and cohomology theory, the former being especially suited for interpretation through the Aharanov-Susskind gedanken experiment. The last two sections are concerned with the topology of spinor space-times. Geroch's (1968) theorem on the parallelisability of non-compact spinor space-times is compared with some of Rohlin's (1958) classic results on four-dimensional spin-manifolds and, in the last section, I define linear spin cobordism and compute the linear spin cobordism of compact spinor space-times.

2. *Spin and Spinor Structures*

2.1. *The Lorentz Groups*

Minkowski space will be denoted by $\mathbb{R}^{1,3}$ and represents the vector space $\mathbb{R}^4$ with the indefinite non-degenerate bilinear form $\langle , \rangle$ of signature $(1, 3)$. The Lorentz group $L = O(1, 3)$ is the group of orthogonal automorphisms of $\mathbb{R}^{1,3}$ and its invariant subgroups include the proper Lorentz group L_+ of orientation preserving automorphisms of $\mathbb{R}^{1,3}$ and the proper orthochronous Lorentz group $L_+^\uparrow$ of automorphisms preserving the semi-orentations of $\mathbb{R}^{1,0}$ and $\mathbb{R}^{0,3}$. The latter two groups will be denoted by $SO(1, 3)$ and $SO_+(1, 3)$ respectively. The unit timelike pseudosphere of $\mathbb{R}^{1,3}$ denoted by $TS^{1,3}$ is defined as the set of all x in $\mathbb{R}^{1,3}$ with $\langle x, x \rangle = 1$; the unit space-like pseudosphere of $\mathbb{R}^{1,3}$ is denoted by $SS^{1,3}$ and is defined as the set of all x in $\mathbb{R}^{1,3}$ with $\langle x, x \rangle = -1$ and if S^3 as usual denoted the unit positive definite sphere in $\mathbb{R}^4$, the unit null pseudosphere in $\mathbb{R}^{1,3}$ is denoted by $NS^{1,3}$ and is defined as the intersection of the non-zero null vectors in $\mathbb{R}^{1,3}$ with S^3. In direct analogy to the projective space $\mathbb{R}P^3$ of lines in $\mathbb{R}^4$ we define the projective spaces of timelike, space-like or null lines in $\mathbb{R}^{1,3}$ denoted respectively by $TP\mathbb{R}^{1,3}$, $SP\mathbb{R}^{1,3}$ and $NP\mathbb{R}^{1,3}$. These are isomorphic respectively to $TS^{1,3}/\mathbb{Z}_2$, $SS^{1,3}/\mathbb{Z}_2$ and $NS^{1,3}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ is the central subgroup of $O(1, 3)$. The pseudospheres are supposed to carry the relative topologies as subspaces of $\mathbb{R}^4$ and are diffeomorphic to respectively $\mathbb{R}^3 \times S^0$, $S^2 \times \mathbb{R}$ and $S^2 \times S^0$ under the maps $f_T: (x, \mathrm{x}) \mapsto (\mathrm{x}, x/|x|)$, $f_T^{-1}: (\mathrm{x}, e) \mapsto (e(1+\mathrm{x}^2)^{1/2}, x)$;

$f_S: (x, \mathrm{x}) \mapsto (\mathrm{x}/\|\mathrm{x}\|, x)$, $f_S^{-1}: (x, t) \mapsto (t, \mathrm{x}(1 + t^2)^{1/2})$ and $f_N: (x, \mathrm{x}) \mapsto (x/\|x\|, \mathrm{x}/|\mathrm{x}|)$, $f_N^{-1}: (\mathrm{x}, e) \mapsto (e\|\mathrm{x}\|, \mathrm{x})$. The projective spaces have the identification topologies. The following lemma will be useful later on. The three-dimensional cross-cap (without boundary) is defined as the cylinder $S^2 \times [0, 1]$ with antipodal points identified in $S^2 \times O$.

Lemma 1. $TP\mathbb{R}^{1,3} \cong \mathbb{R}^3$, $NP\mathbb{R}^{1,3} \cong S^2$ and $SP\mathbb{R}^{1,3}$ is a three-dimensional cross-cap without boundary.

Proof. There are imbeddings $TP\mathbb{R}^{1,3}$, $SP\mathbb{R}^{1,3}$ and $NP\mathbb{R}^{1,3} \to \mathbb{R}P^3$ defined by projecting the two representative points of either a time-like, space-like or null line onto the two points of intersection of the line with S^3. For example, in Fig. 1, the time-like line represented by the two points A, B, of intersection with $TS^{1,3}$ is represented in $\mathbb{R}P^3$ by the two points A', B' of

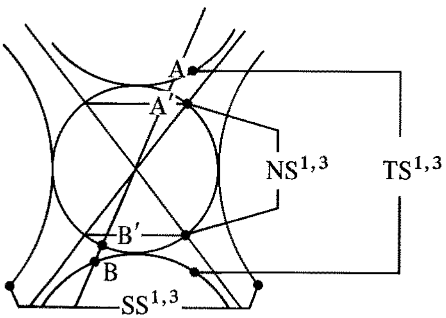


Figure 1.

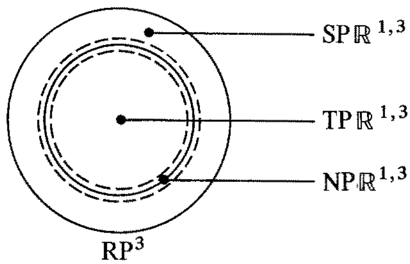


Figure 2.

intersection of the line with S^3. In this way, if we represent $\mathbb{R}P^3$ as a solid three disc of unit radius with antipodal points identified in the boundary S^2, it is immediate that $TP\mathbb{R}^{1,3}$ is imbedded as the open disc of radius $\frac{1}{2}$, $NP\mathbb{R}^{1,3}$ is imbedded as the two-sphere of radius $\frac{1}{2}$ and $SP\mathbb{R}^{1,3}$ as the boundaryless three-dimensional cross-cap. (The interesting point about this is that the time-like projective space is contractible whilst the spaces $SP\mathbb{R}^{1,3}$ and $NP\mathbb{R}^{1,3}$ are not; $SP\mathbb{R}^{1,3}$ is obviously homotopy equivalent to the space $\mathbb{R}P^2$.)

2.2. *Spinor Groups*

The spinor groups associated with the Lorentz groups are subgroups of the group of units of the universal Clifford algebra of the pseudo-orthogonal space $\mathbb{R}^{1,3}$ denoted by $\mathbb{R}_{1,3}$ (Porteous, 1969). $\mathbb{R}_{1,3}$ is isomorphic to the quotient of the tensor algebra of $\mathbb{R}^{1,3}$ modulus the relation $x^2 = -\langle x, x\rangle$ for $x \in \mathbb{R}^{1,3}$. Because the tensor algebra is $\mathbb{Z}_2$-graded into tensors of even or odd degree, so is $\mathbb{R}_{1,3}$ and its components are $\mathbb{R}_{1,3}^0$ and $\mathbb{R}_{1,3}^1$. It can be shown that $\mathbb{R}_{1,3}$ is isomorphic to the algebra of endomorphisms of the two-dimensional quaternionic vector space $\mathbb{H}^2$: $\mathrm{End}_{\mathbb{H}}(\mathbb{H}^2)$ ($\mathbb{H}$ for Hamilton, denotes the division ring of quaternions). Because of this, $\mathbb{H}^2$ is called the spinor space of $\mathbb{R}^{1,3}$. The subalgebra $\mathbb{R}_{1,3}^0$ can be shown to be isomorphic to

$\mathbb{R}_{1,2}$ which is itself isomorphic to the endomorphism algebra $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$. The latter space is the spinor space used in physics (for a reason which will become clear later in this section). The Clifford group $\Gamma(1,3)$ of $\mathbb{R}^{1,3}$ is defined as a certain subgroup of the group of units of $\mathbb{R}_{1,3}$ and has the index two invariant subgroup $\Gamma^0(1,3)$ defined as $\Gamma(1,3) \cap \mathbb{R}^0_{1,3}$. There is a group epimorphism $p : \Gamma(1,3) \to O(1,3)$ defined by regarding $\mathbb{R}^{1,3}$ as a linear subspace of $\mathbb{R}_{1,3}$ and writing $p(g) : x \mapsto \hat{g}.x.g^{-1}$ for $g \in \Gamma(1,3)$ and $x \in \mathbb{R}^{1,3}$. The kernel of p is isomorphic to the group $GL(1)$ of non-zero real numbers which contains the subgroup of non-zero positive numbers $GL_+(1)$. The quotient group $\Gamma(1,3)/GL_+(1)$ is denoted by $\mathrm{Pin}(1,3)$ and there is an obvious epimorphism $\mathrm{Pin}(1,3) \to O(1,3)$ with kernel $\mathbb{Z}_2$ generated by the coset of $-1 \in GL(1)$. It is therefore clear that $\mathrm{Pin}(1,3)$ is a Lie group and that $\mathrm{Pin}\,(1,3) \to O(1,3)$ is a two-fold covering space. The subgroup $\Gamma^0(1,3)/GL_+(1)$ of $\mathrm{Pin}(1,3)$ is invariant of index two and is called $\mathrm{Spin}(1,3)$. The restriction of the epimorphism $p : \mathrm{Pin}(1,3) \to O(1,3)$ sends $\mathrm{Spin}(1,3)$ onto $SO(1,3)$. By defining a 'norm' homomorphism $N : \mathbb{R}_{1,3} \to \mathbb{R}$, $N : x \mapsto x.\bar{x}$, one can show that $\mathrm{Pin}(1,3)$ and $\mathrm{Spin}(1,3)$ are isomorphic to the subgroups $\ker(|N|)$ of $\Gamma(1,3)$ respectively $\Gamma^0(1,3)$. The group $\mathrm{Spin}_+(1,3)$ is defined as $\ker(N)$ and is an index two invariant subgroup of $\mathrm{Spin}(1,3)$. $\mathrm{Spin}_+(1,3)$ is the component of the identity of $\mathrm{Pin}(1,3)$ and is therefore mapped onto the component of the identity $SO_+(1,3)$ by the projection p. Because $\mathbb{R}^0_{1,3}$ is isomorphic to $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$, $\mathrm{Spin}_+(1,3)$ maps isomorphically onto $SL(2,\mathbb{C})$. This is why $\mathbb{C}^2$ is called spinor space by physicists.

We shall need to consider the orthogonal inclusion $\mathbb{R}^{0,3} \to \mathbb{R}^{1,3}$ which induces an algebra homomorphism $\mathbb{R}_{0,3} \to \mathbb{R}_{1,3}$ preserving $\mathbb{Z}_2$-degree, $\mathbb{R}^0_{0,3} \to \mathbb{R}^0_{1,3}$. The algebra homomorphism induces a monomorphism $\mathrm{Spin}(0,3) \to \mathrm{Spin}_+(1,3)$. Because there are isomorphisms $\mathbb{R}^0_{0,3} \cong \mathbb{R}^0_{3,0}$, $\mathrm{Spin}(0,3) \cong \mathrm{Spin}(3,0) := \mathrm{Spin}(3)$. There is an analogous epimorphism $p' : \mathrm{Spin}(3) \to SO(3)$ with kernel $\mathbb{Z}_2$ such that the following diagram commutes:

$$(*)\qquad \begin{array}{ccccc} \mathbb{Z}_2 & \xrightarrow{\text{inc.}} & \mathrm{Spin}(0,3) & \xrightarrow{p'} & SO(3) \\ \| & J_1 & \downarrow \text{inc.} & J & \downarrow \text{inc.} \\ \mathbb{Z}_2 & \xrightarrow[\text{inc.}]{} & \mathrm{Spin}_+(1,3) & \xrightarrow[p]{} & SO_+(1,3) \end{array}$$

Because $\mathbb{R}^0_{0,3} \cong \mathbb{R}^0_{3,0} \cong \mathbb{R}_{2,0} \cong \mathbb{H}$, $\mathrm{Spin}(0,3)$ can be identified with the group $\mathrm{Sp}(1)$ of unit quaternions. Therefore $\mathrm{Spin}(0,3)$ is topologically S^3. (Note that the group $SU(2)$ is isomorphic to $\mathrm{Sp}(1)$.) We shall need the following lemmas on the topology of the Lorentz and spinor groups.

Lemma 2. $SO_+(1,3) \cong SO(3) \times \mathbb{R}^3$, $\mathrm{Spin}_+(1,3) \cong \mathrm{Spin}(3) \times \mathbb{R}^3$ and $SO(3) \cong \mathbb{R}P^3$.

Proof. (a) Consider the principal $SO(3)$-bundle defined by the projection $SO_+(1,3) \to SO_+(1,3)/SO(3)$. Because $SO_+(1,3)$ acts transitively on $TS^{1,3}_+$

with stability group $SO(3)$, $SO_+(1,3)/SO(3) \cong TS_+^{1,3} \cong \mathbb{R}^3$. Therefore since $\mathbb{R}^3$ is contractible, the principal bundle is trivial: $SO_+(1,3) \cong SO(3) \times \mathbb{R}^3$. (b) Because $\text{Spin}_+(1,3)$ is the simply connected two-fold cover of $SO_+(1,3)$ it must be isomorphic to the simply connected two-fold cover of $SO(3) \times \mathbb{R}^3$ which is $\text{Spin}(3) \times \mathbb{R}^3$. (c) $SO(3) \cong \text{Spin}(3)/\mathbb{Z}_2 \cong S^3/\mathbb{Z}_2 = \mathbb{R}P^3$.

Lemma 3. The homotopy group $\pi_1(SO(3), e)$ is generated over $\mathbb{Z}_2$ by the homotopy class of any parameterised rotation through 2π about any axis regarded as a loop $\sigma : (S^1, e) \to (SO(3), e)$.

Proof. Because $\text{Spin}(3) \to SO(3)$ is a two-fold covering space of $SO(3)$ and Spin(3) is simply connected, $\pi_1(SO(3), e)$ is isomorphic to the deck group $\mathbb{Z}_2$ generated by the antipodal map $x: \mapsto -x$ on Spin(3). The isomorphism is induced by sending the antipodal map onto the homotopy class of the *loop* $p \circ w$ where w is any path in Spin(3) from e to $-e$. But such a path is represented by any map of the form (say) $w : t \mapsto \cos(\pi t) + i \sin(\pi t)$ for $0 \leqslant t \leqslant 1$ (we regard Spin(3) as Sp(1) and $\mathbb{R}^3 \subset \mathbb{H}$ as the space of purely imaginary quaternions). The loop $p \circ w$ defines a parameterised rotation through 2π about the x-axis in $\mathbb{R}^3$.

Corollary. $\pi_1(SO_+(1,3), e)$ is generated by the homotopy class of any parameterised rotation through 2π in $\mathbb{R}^{0,3}$.

Proof. It follows from Lemma 2 that the inclusion homomorphism $SO(3) \overset{J}{\to} SO_+(1,3)$ is a homotopy equivalence. Therefore the induced homotopy homomorphism $J_*: \pi_1(SO(3), e) \to \pi_1(SO_+(1,3), e)$ is an isomorphism.

Lemma three and its corollary are vital for an understanding of the interpretation of spinor structures on space-times from the homotopy point of view.

2.3. *Lorentzian Structures*

If X is a four-manifold, its tangent bundle will be denoted by $t_X = \pi_X: T(X) \to X$. The principal frame bundle of t_X will be called the Einstein bundle of X and written $GL(4)(X)$, where $GL(4)$ is, in this context, the Einstein group. We shall need the notion of homomorphism of principal bundles. Suppose that $b_1 = p_1 : E_1 \to X$ and that $b_2 = p_2 : E_2 \to X$ are principal G_1 respectively G_2 bundles over X where G_1 and G_2 are Lie groups with actions $a_1 : E_1 \times G_1 \to E_1$ respectively $a_2 : E_2 \times G_2 \to E_2$. Then a principal bundle homomorphism $f: b_1 \to b_2$ is defined by a homomorphism $f_1 : G_1 \to G_2$ and a smooth map $f_2 : E_1 \to E_2$ such that the following diagram commutes:

$$
\begin{array}{ccccc}
E_1 \times G_1 & \xrightarrow{a_1} & E_1 & & \\
\Big\downarrow f_2 \times f_1 & & \Big\downarrow f_2 & \searrow^{p_1} & \\
 & & & & X \\
E_2 \times G_2 & \xrightarrow[a_2]{} & E_2 & \nearrow_{p_2} &
\end{array}
$$

If f_1 is a smooth subgroup monomorphism and f_2 a smooth submanifold then the homomorphism is called a smooth reduction of the structural group G_2 to G_1. If f_1 is a smooth epimorphism and f_2 is a fibration, then f is called a smooth extension of the structural group G_2 of b_2 to G_1 (in this case, G_1 is a group extension of G_2 by $\ker(f_1)$).

A Lorentzian structure on a four-manifold X is a reduction of the Einstein group $GL(4)$ of the principal frame bundle of X to the Lorentz group $O(1, 3)$. Such a reduction is possible iff there exists a smooth global section of the fibre bundle $GL(4)/O(1, 3)(X)$ with fibre $GL(4)/O(1, 3)$ associated with the Einstein bundle. But the space $GL(4)/O(1, 3)$ is in one-to-one correspondence with the Lorentz signature bilinear forms on $\mathbb{R}^4$. Therefore a Lorentzian structure on X: the smooth assignment of the inertial frames in the Einstein bundle is equivalent to finding a smooth Lorentz tensor on X or putting a smooth $\mathbb{R}^{1,3}$-structure on each tangent space of X. Any four-manifold admitting a smooth Lorentzian structure is called a space-time. Steenrod's (1951) theorem sets up a one-to-one correspondence between Lorentzian structures on a manifold and tangent line-bundles. The correspondence is obtainable in the following way. Suppose that X admits a Lorentzian structure. Then we may form the associated bundle of time-like one-dimensional subspaces of t_X with fibre $TP\mathbb{R}^{1,3}$. But we have already noted that $TP\mathbb{R}^{1,3}$ is contractible. Thus by a standard result in the theory of fibre bundles (Husemoller, 1966) there is a global section of the $TP\mathbb{R}^{1,3}$ bundle and hence a smooth (time-like) tangent line-bundle on X. Conversely, because $\mathbb{R}P^3 \cong O(4)/O(1) \times O(3)$ and $O(4)$ is a subgroup of $GL(4)$ and $O(1) \times O(3) = O(4) \cap O(1, 3)$ is a subgroup of $O(1, 3)$, $\mathbb{R}P^3$ is a subspace of $GL(4)/O(1, 3)$. Hence a global section of the projective bundle of t_X: a tangent line-bundle defines a section of the bundle $GL(4)/O(1, 3)(X)$ and therefore a Lorentzian structure on X. A slight extension of the Poincaré-Hopf theorem (a compact manifold admits a tangent line-bundle iff it has Euler number zero) gives the well-known result that a compact closed four-manifold can admit a Lorentzian structure iff it has Euler number zero. Any non-compact four-manifold admits a tangent line-bundle and therefore a Lorentzian structure. A Lorentzian structure is called orientable iff its structural group $O(1, 3)$ reduces to the $SO(1, 3)$. It is clear that such a reduction can be performed iff X is an orientable four-manifold. A Lorentzian structure is called time-orientable iff its structural group reduces to the orthochronous Lorentz group $L\uparrow$ of orthogonal automorphisms of $\mathbb{R}^{1,3}$ which preserve the orientation of $\mathbb{R}^{1,0}$. Such a reduction can be performed iff there is a lifting of the section of the $TP\mathbb{R}^{1,3}$-bundle to its two-fold cover the $TS^{1,3}$-bundle or iff the two-fold covering space $NS^{1,3}(X) \to NP\mathbb{R}^{1,3}(X)$ is trivial. Lastly a Lorentzian structure is called space and time orientable iff its structural group reduces to the group $SO_+(1,3)$, iff X is orientable and time-orientable. (In this case if e_0^1 is the trivial line-bundle induced by the reduction to L, and η is the complementary space-like three-plane bundle with $t_X = e_0^1 \oplus \eta$, η is an orientable vector bundle.) The structural group $SO_+(1, 3)$ of a space and time-orientable space-time always further reduces to $SO(3)$ the structural group of η because $SO_+(1, 3)/SO(3)$

is contractible. This is important in the following because one can always replace the group $SO_+(1, 3)$ by $SO(3)$ in the calculations and use well-known results on $SO(3)$-bundles.

2.4. *Spinor-Structures*

A spinor-structure over a Lorentzian structure on a space-time X is an extension of the Lorentz structural group to a spinor group. That is of $O(1, 3)$, $SO(1, 3)$ or $SO_+(1, 3)$ to respectively Pin$(1, 3)$, Spin$(1, 3)$ or Spin$_+(1, 3)$. From our earlier definition of the extension of a structural group, it follows that one can find such an extension iff there exists a principal spin-bundle and a two-fold covering map from the spin-bundle onto the Lorentz bundle such that the restriction of the covering to each fibre of the spin-bundle over X coincides with the two-fold covering map Pin$(1, 3) \to O(1, 3)$ respectively Spin$(1, 3) \to SO(1, 3)$ or Spin$_+(1, 3) \to SO_+(1, 3)$. Such a covering of the Lorentz bundle is called a spin-structure on X. The associated spinor-bundles are defined via the representations of Pin$(1, 3)$ in $\mathbb{H}^2$ and Spin$(1, 3)$ or Spin$_+(1, 3)$ in $\mathbb{C}^2$. Local sections of either the symplectic two-plane bundle or the complex two-plane bundle of a spinor space-time are called spinor fields on X. Their physical interpretation is in terms of the wave-functions of spin $\frac{1}{2}$ fermions: neutrinos, electrons, kaons . . ..

For a paracompact manifold X there is a one-to-one correspondence between the set of isomorphism classes of $O(n)$-bundles on X and homotopy classes of maps into a universal classifying space for $O(n)$-bundles $BO(n)$ (the direct limit of Grassmannians $G_n(\mathbb{R}^{n+s})$ over s). The correspondence is $[X, BO(n)]^* \to$ isomorphism classes of principal $O(n)$-bundles on X; $|f| \mapsto f^*(b_n)$ the pull-back along $f: X \to BO(n)$ of the universal principal $O(n)$ bundle b_n over $BO(n)$ (the direct limit of the principal $O(n)$-bundles $SV_n(\mathbb{R}^{n+s}) \to G_n(\mathbb{R}^{n+s})$ over s). At this point we shall only be interested in $BO(1) = B\mathbb{Z}_2$. Then there is a one-to-one correspondence between the equivalence classes of two-fold coverings of X and the homotopy set $[X, B\mathbb{Z}_2]$. It can be shown that $B\mathbb{Z}_2$ is an Eilenberg-MacLane space (Spanier, 1966) of type $K(\mathbb{Z}_2, 1)$, that is $\pi_q(B\mathbb{Z}_2) \cong \mathbb{Z}_2$ if $q = 1$ or 0 otherwise. Thus the set $[X, B\mathbb{Z}_2]$ is isomorphic with $[X, K(\mathbb{Z}_2, 1)]$. But the latter set is the generalised cohomology group $H^1(X, \underset{\sim}{K}\mathbb{Z}_2)$ of X in the spectrum $\underset{\sim}{K}\mathbb{Z}_2$ (Hilton, 1971). The Eilenberg-Steenrod theorem shows that the former cohomology group is isomorphic to the usual singular mod(2) group. The correspondence can be regarded as sending a principal $\mathbb{Z}_2$-bundle into its first Stiefel-Whitney class. One can also find a one-to-one correspondence between principal $\mathbb{Z}_2$-bundles over a connected space X and the group $\pi_1(X, x_0) \pitchfork \mathbb{Z}_2$ of homomorphisms of the (non-abelian in general) group $\pi_1(X, x_0)$ into $\mathbb{Z}_2$. Suppose that a $\mathbb{Z}_2$-bundle over X is classified by a map $f: (X, x_0) \to (B\mathbb{Z}_2, f(x_0))$. Then there is an induced homotopy homomorphism $f_*: \pi_1(X, x_0) \to \pi_1(B\mathbb{Z}_2, f(x_0)) \cong \mathbb{Z}_2$. If $w \in \pi_1(X, x_0)$ is the homotopy class of a loop $w: (S^1, e) \to (X, x_0)$ in X at x_0, then $f_*(|w|) = |f o w|$ which is the class of the loop $f o w$ in $B\mathbb{Z}_2$ at

$f(x_0)$. fow is the classifying map of the principal $\mathbb{Z}_2$-bundle $w^*(f^*(b_1))$ on S^1 which is the pull-back to S^1 along w of the principal $\mathbb{Z}_2$-bundle $f^*(b_1)$ on X. Therefore $f^*(b_1)$ is trivial if it is trivial along any imbedded circle in X and is non-trivial iff there is an imbedded circle along which it is the non-trivial two-fold cover of S^1. Equivalently the homomorphism associated with principal $\mathbb{Z}_2$-bundle on a connected X assigns the value 1 to a loop in X based at some x_0 if the loop lifts to a loop in the covering space and -1 to a loop that lifts to an open path in the covering space (Corresponding to the permutation of the fibre over x_0 induced by the unique lift of the loop from a point in the fibre: either the identity permutation or the interchange permutation.) Finally the first Stiefel-Whitney class of a principal $\mathbb{Z}_2$-bundle over a space X is defined as follows. The cohomology algebra $H^*(B\mathbb{Z}_2, \mathbb{Z}_2)$ is the polynomial algebra on the universal Stiefel-Whitney class W_1 the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$. If $f: X \to B\mathbb{Z}_2$ is the classifying map of a principal $\mathbb{Z}_2$-bundle on X, then the class $f^*(W_1) \in H^1(X, \mathbb{Z}_2)$ is the first Stiefel-Whitney class of $f^*(b_1)$ which is trivial iff the bundle is trivial.

Because a spin-structure on a space-time is a certain two-fold over of the Lorentz bundle, one can formulate the following definition of a spinor-structure (Milnor, 1963). A spinor-structure on X is a cohomology class $S \in H^1(L(X), \mathbb{Z}_2)$ (where $L(X)$ is the Lorentz bundle of X) such that for any $x \in X$, the fibre inclusion $i_x: L_x \to L(X)$ via $i_x^*: H^1(L(X), \mathbb{Z}_2) \to H^1(L_x, \mathbb{Z}_2)$ sends S into s_x the first Stiefel-Whitney class of the two-fold cover $\mathrm{spin}_x \to L_x$. We are therefore interested in the cohomology groups $H^1(L, \mathbb{Z}_2)$ where L is either of $O(1,3)$, $SO(1,3)$ or $SO_+(1,3)$. Note that $O(1,3)$ is topologically $SO_+(1,3) \times S^0 \times S^0$ and that $SO(1,3)$ is topologically $SO_+(1,3) \times S^0$.

Lemma 4. $H^1(SO_+(1,3), \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$ with generator the first Stiefel-Whitney class of the $\mathbb{Z}_2$-bundle $\mathrm{Spin}_+(1,3) \to SO_+(1,3)$.

Proof. We shall prove the slightly more useful result that $H^1(SO(3), \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$ with generator the Stiefel-Whitney class of the $\mathbb{Z}_2$-bundle $\mathrm{Spin}(3) \to SO(3)$. Recall that we can regard $\mathrm{Spin}(3) \to SO(3)$ as the cover $S^3 \to \mathbb{R}P^3$. The first Stiefel-Whitney class of the latter is non-trivial and can be taken as the generator. (It is the Stiefel-Whitney class of the canonical line bundle over $\mathbb{R}P^3$ with total space $\mathbb{R}^4 - O$.) Therefore because the inclusions $\mathrm{Spin}(3) \to \mathrm{Spin}_+(1,3)$ and $SO(3) \to SO_+(1,3)$ commute with the covering maps and are homotopy equivalences the result follows.

Corollary. $H^1(SO(1,3), \mathbb{Z}_2) \cong \mathbb{Z}_2(w_1)^2$ and $H^1(O(1,3), \mathbb{Z}_2) \cong \mathbb{Z}_2(w_1)^4$ where in each case w_1 is the Stiefel-Whitney class of $\mathrm{Spin}_+(1,3) \to SO_+(1,3)$.

Proof. One can regard the coverings $\mathrm{Spin}(1,3) \to SO(1,3)$ and $\mathrm{Pin}(1,3) \to O(1,3)$ as respectively two and four copies of $\mathrm{Spin}_+(1,3) \to SO_+(1,3)$ on the connected components of the spinor-groups, $\mathrm{Pin}(1,3)$ being topologically $\mathrm{Spin}_+(1,3) \times S^0 \times S^0$ and $\mathrm{Spin}(1,3)$ being $\mathrm{Spin}_+(1,3) \times S^0$.

We shall need the following lemma which reduces the calculation of the $\mathrm{Spin}_+(1,3)$-structures on the principal $SO_+(1,3)$-bundle of a space-time to the calculation of the Spin(0, 3)-structures on the reduction of the Lorentz bundle to $SO(3)$.

Lemma 5. An $SO_+(1,3)$-structure extends to a $\mathrm{Spin}_+(1,3)$-structure iff the $\mathrm{Spin}_+(1,3)$-structure reduces to a Spin(0, 3)-structure covering the canonical reduction of the $SO_+(1,3)$-structure to $SO(3)$.

Proof. The quickest way to see this is by looking at the Čech cohomology sets (Hirzebruch, 1966) $\check{H}^1(X, \mathbf{G})$ of X in the constant presheaf $\mathbf{G}$ on X. Then corresponding to the commutative diagram (*) there is a Čech cohomology ladder of Bockstein coefficient sequences, part of which is:

$$\begin{array}{ccccc} \longrightarrow & \check{H}^1(X, \mathbf{Spin}(0,3)) & \xrightarrow{p'_*} & \check{H}^1(X, \mathbf{SO}(3)) & \longrightarrow \\ & \downarrow{\scriptstyle J'_*} & & \downarrow{\scriptstyle J_*} & \\ \longrightarrow & \check{H}^1(X, \mathbf{Spin}_+(1,3)) & \xrightarrow[p_*]{} & \check{H}^1(X, \mathbf{SO}_+(1,3)) & \rightarrow \end{array}$$

The vertical arrows are ismorphisms because the inclusion maps J, J' are homotopy equivalences. A principal bundle class $w \in \check{H}^1(X, \mathbf{SO}_+(1,3))$ extends to $\mathrm{Spin}_+(1,3)$ iff $w \in \mathrm{Im}(p_*)$ iff its canonical reduction to $SO(3): J_*^{-1}(w)$ lies in $\mathrm{Im}(p'_*)$, i.e. extends to Spin(0, 3) as the restriction of the $\mathrm{Spin}_+(1,3)$-structure to Spin(0, 3).

The problem of finding necessary and sufficient conditions for a spin-structure to exist on an $SO(n)$-bundle is solved by the following theorem (Milnor, 1963) (see also Bichteler, 1967), which also calculates the number of Spin(n)-structures (if any) on an $SO(n)$-bundle.

Theorem. A principal $SO(n)$-bundle on a compact manifold X extends to a Spin(n)-structure iff its second Stiefel-Whitney class $w_2 \in H^2(X, \mathbb{Z}_2)$ is trivial. If $w_2 = 0$, then the Spin(n)-structures covering the $SO(n)$ are in one-to-one correspondence with the group $H^1(X, \mathbb{Z}_2)$.

Proof. Suppose that $b = p : E \to X$ is a principal $SO(n)$-bundle on X. Then because the group $SO(n)$ is path connected, the fibration is orientable (Bichteler, 1967). (That is, the homotopy group has a trivial action on the fibres.) This means that one can extract the following exact sequence from the spectral sequence of b and the universal $SO(n)$-bundle $b'_n = p_n : E'_n \to BSO(n)$:

$$\begin{array}{ccccc}
O & \longrightarrow & H^1(X,\mathbb{Z}_2) & \xrightarrow{p^*} & H^1(E,\mathbb{Z}_2) \\
 & & \uparrow{\scriptstyle \bar{f}^*} & & \uparrow \\
O & \xrightarrow{f^*} & H^1(BSO(n),\mathbb{Z}_2) & \xrightarrow[p_n^*]{} & H^1(E'_n,\mathbb{Z}_2)
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{i^*} & H^1(SO(n),\mathbb{Z}_2) & \xrightarrow{d^*} & H^2(X,\mathbb{Z}_2) & \longrightarrow \\
 & \| & & {\scriptstyle f^*}\uparrow & \\
\xrightarrow[i_n^*]{} & H^1(SO(n),\mathbb{Z}_2) & \xrightarrow[d_n^*]{} & H^2(BSO(n),\mathbb{Z}_2) & \to
\end{array}$$

$f : X \to BSO(n)$ is the classifying map of $b : b \cong f^*(b_n)$. We need the following facts (i) $H^*(BSO(n), \mathbb{Z}_2)$ is isomorphic to the $\mathbb{Z}_2$ polynomial algebra with generators the universal Stiefel-Whitney classes W_i for $i \geqslant 1$ modulus the ideal generated by W_1; therefore $H^2(BSO(n), \mathbb{Z}_2)$ is generated by the universal Stiefel-Whitney class W_2. (ii) $H^1(E'_n, \mathbb{Z}_2) = 0$ because E'_n has $\pi_k(E'_n) = 0$ for all $k \geqslant 0$. In the diagram, the homomorphism i^* is induced by the inclusion of a typical fibre into the total space of the bundle. Suppose that $w \in H^1(SO(n), \mathbb{Z}_2)$ is the generator of the group and represents the first Stiefel-Whitney class of the $\mathbb{Z}_2$-bundle Spin$(n) \to SO(n)$. Then b is covered by a Spin(n)-bundle iff $w \in \text{Im}(i^*)$ iff $w \in \text{Ker}(d^*)$ iff $f^* o d_n^*(w) = 0$. But because $H^1(E'_n, \mathbb{Z}_2) = 0$, d_n^* is a monomorphism and therefore takes the generator w of $H^1(SO(n), \mathbb{Z}_2)$ into the generator W_2 of $H^2(BOS(n), \mathbb{Z}_2)$. Hence there is a Spin$(n)$-structure on b iff $f^*(W_2) = w_2(f^* b_n) = w_2(b) = 0$. Suppose that $w_2(b) = 0$, then the Spin(n)-structures covering b are in one-to-one correspondence with the $W \in H^1(E, \mathbb{Z}_2)$ mapped into w by i^*: therefore in one-to-one correspondence with ker(i^*) (any two differ by an element of the kernel) and hence with Im$(p^*) \cong H^1(X, \mathbb{Z}_2)$.

Corollary. A space-time X with $SO_+(1, 3)$ structural group admits a Spin$_+(1, 3)$-structure iff $w_2(t_X) = 0$.

Proof. By Lemma 5 the space-time admits a Spin$_+(1, 3)$-structure iff its reduction to $SO(3)$ admits a Spin(3)-structure iff $w_2(\eta) = w_2(t_X) = 0$.

The formulation of spinor-structures in homotopy theoretic terms is the most convient one to lead to a physical interpretation (Clarke, 1971). Recall that the set of isomorphism classes of principal $\mathbb{Z}_2$-bundles on a space X is in one-to-one correspondence with $H^1(X, \mathbb{Z}_2) = [X, B\mathbb{Z}_2]$ and that the mapping $|f| \in [X, B\mathbb{Z}_2] \mapsto f_* \in \pi_1(X, x_0) \pitchfork \mathbb{Z}_2$ establishes a further correspondence with $\pi_1(X, x_0) \pitchfork \mathbb{Z}_2$. The latter correspondence is natural, therefore if $i_x: SO_+(1,3)_{x_0} \to SO_+(1,3)(X)$ is the inclusion of a typical fibre into the total space of the Lorentz bundle, the following diagram commutes:

$$\begin{array}{ccc} H^1(SO_+(1,3)(X), \mathbb{Z}_2) & \xrightarrow{I_X} & \pi_1(SO_+(1,3)(X), b) \pitchfork \mathbb{Z}_2 \\ \Big\downarrow i_x^* & & \Big\downarrow i_x^* \\ H^1(SO_+(1,3)_x, \mathbb{Z}_2) & \xrightarrow[I_x]{} & \pi_1(SO_+(1,3)_x, b) \pitchfork \mathbb{Z}_2 \end{array}$$

where I denotes the natural isomorphisms. Therefore there exists a class $W \in H^1(SO_+(1,3)(X), \mathbb{Z}_2)$ with $i_x^*(W) = w_x$ the first Stiefel-Whitney class of $\text{Spin}_+(1,3) \to SO_+(1,3)$ iff there exists a homomorphism $\pi_1(SO_+(1,3), b) \xrightarrow{\Gamma} \mathbb{Z}_2$ such that $i_x^* \pitchfork 1(\Gamma) = \Gamma o i_x^* = \Gamma$, where Γ_x is the characteristic homomorphism for $\text{Spin}_+(1,3)_x \to SO_+(1,3)_x$ defined as 1 respectively -1 on a loop class $|\sigma|$ according to whether σ lifts to a loop or an open curve in $\text{Spin}_+(1,3)$. Because a 2π rotation around any axis in $\mathbb{R}_x^{0,3}$ lifts to an open curve from e to $-e$, Γ_x is defined by sending a generator into -1. (Note that in this case the homomorphisms i_x^* are momomorphisms so that no 2π rotation is null homotopic in the whole frame bundle.)

The Aharanov-Susskind gedanken experiment (Aharanov & Susskind, 1967; Hegerfeld & Kraus, 1968) is a theoretical design for an apparatus which could realise the characteristic homomorphism Γ of a spinor-structure on space-time. The apparatus would consist of two halves each containing some electronic system, the two halves being separated by a barrier through which there is a tunnelling current proportional to $\sin(\alpha)$ where α is the relative phase of the electronic wave functions for each half of the apparatus. If the two halves are fitted together there is a current of +1 through the barrier. The two halves are then separated and one-half rotated through 2π relative to the other in some frame. Because the electronic wave functions transform under the basic Spin(3) representation in $\mathbb{C}^2$, corresponding to a projective representation of $SO(3)$, a 2π rotation changes the phase by a factor -1. Thus when the apparatus is refitted together in its original configuration, the electronic wave functions in the two halves of the apparatus have relative phase $-\alpha$ and a current of -1 flows in the barrier. Therefore via the above procedure the homomorphism Γ_x is realised at a space-time point x. A homomorphism $\pi_1(SO_+(1,3)(X), b) \to \mathbb{Z}_2$ can be constructed by the transport of the Aharanov-Susskind apparatus around loops in space-time based at the point under the basis b, keeping to the frame b and therefore describing loops in $SO_+(1,3)$ based at b. More precisely, the apparatus is first fitted together, the tunnelling current is measured and the two halves are separated and then one-half transported about one of the above loops in X. When the two halves are recombined the sign of the tunnelling current measures the number of 2π relative rotations mod(2), defining $\Gamma(|\sigma|) = \pm 1$

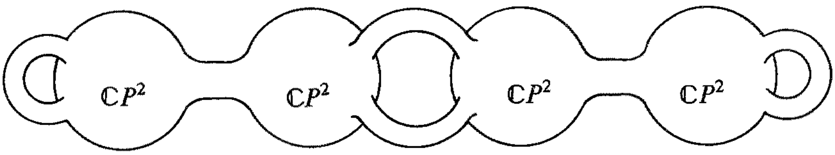
according to the mod(2) number of relative 2 rotations experienced along σ. Γ obviously restricts to Γ_x in each fibre.

2.5. *Topology of Spinor Space-Times*

We next consider some results on the topology of spinor space and time-orientable space-times. Geroch (1968) proved that if a non-compact space and time-orientable space-time admits a $\text{Spin}_+(1, 3)$-structure it must be the trivial one, that is, there must be a global section of the principal $\text{Spin}_+(1, 3)$-bundle covering the Lorentz bundle $SO_+(1, 3)(X)$. Therefore by projecting the section onto the Lorentz bundle via the spin covering map, one obtains a global section of the Lorentz bundle defining a parallelisation of the space-time. Geroch's method was essentially to generalise the usual obstruction theory for compact manifolds to non-compact manifolds. The obstruction to extending a cross-section off the q-skeleton of a CW-complex to the $q + 1$ skeleton can be represented as a cohomology class of $H^q(X, \pi_{q-1}(F))$ where F is the fibre of the fibration. Therefore for $F = \text{Spin}_+(1, 3)$, which is homotopy equivalent to S^3, the only obstruction to finding a global section is in $H^4(X, \pi_3(S^3)) \cong H^4(X, \mathbb{Z})$. If X is oriented and non-compact, $H^4(X, \mathbb{Z}) = 0$, so that there is no obstruction to a section. (Incidentally, this is just the method used to demonstrate that any compact, oriented three-manifold is parallelisable. For by a theorem of Wu if X is a compact oriented manifold of dimension $\equiv 3$ mod.(4), $w_{n-1} = 0$. Therefore for a three-manifold, $w_2 = 0$ and the principal $SO(3)$ frame manifold admits a Spin(3)-structure. Thus because S^3 is 2-connected, there is a global section of the spin-structure and hence a parallelisation. There is a similar theorem for oriented even dimensional compact manifolds (Massey, 1960). Thus for any compact space and time oriented space-time $w_3 = 0$.)

A related result for compact spinor space-times is contained in the proof of a classic theorem of Rohlin (Milnor & Kervaire, 1958; Rohlin, 1958). Rohlin proved that any compact, orientable four-dimensional spin-manifold and any point x of the manifold, the open submanifold X-x is parallelisable, that is, the manifold X is 'almost' parallelisable. The obstruction to extending the section from X-x to all of X is defined as the homotopy class of the restriction of the section to any boundary three sphere of a closed four-disc (containing the point x in its interior) regarded as a map $S^3 \to SO(4)$. That is, the obstruction is an element of $\pi_3(SO(4))$. One can regard X as imbedded in some R^{m+4} for m large. That X is almost parallelisable means that for any $x \in X$, the normal bundle N of X in R^{m+4} is trivial when restricted to X-x. As above, the obstruction to extending the trivialisation to all of X is an element of the group $\pi_3(SO(m))$. There is a homomorphism $J : \pi_3(SO(m)) \to \pi_{m+3}(S^m)$ and the obstruction (which is an integer) can be shown to lie in ker(J). For large enough m, it can be shown that $\pi_{m+3}(S^m)$ is isomorphic to $\mathbb{Z}_{24}$. Therefore the integer, which is shown to be half the Pontryagin number $P_1\langle X\rangle$ of X, is divisible by 24. And $P_1\langle X\rangle$ is divisible by 48. By a further result of Rohlin and Thom (a special case of the Hirzebruch signature theorem (Hirzebruch, 1966)) the topological signature, sig(X), of X is divisible by

16. The latter result enables one to construct examples of oriented, time-oriented space-times with no spinor-structure (since there can be no Spin(4)-structure on the principal $SO(4)$ frame bundle). The easiest examples come from $2k$ copies of $\mathbb{C}P^2$ for $k < 8$ which has signature $2k$ and Euler number $6k$. By adding $2k - 1$ one handles to connected the manifold one reduces the Euler number to $2k + 2$. By adding a further $2k + 1$ one handles the Euler number is reduced to zero and signature is an oriented cobordism invariant and spherical modification preserves oriented cobordism class, the modified manifold is a space-time (trivial Euler number) of signature $2k \not\equiv 0 \bmod(16)$ and can therefore admit no spinor structure.



2.6. *The Spin-Cobordism of Compact Space-Times*

In Whiston (1974) I noted that two compact, closed, oriented space-times are cobordant in the oriented sense (Strong, 1969) iff they have the same topological signature, therefore because the forgetful homomorphism from spin-cobordism classes (Anderson *et al.* 1967) to oriented cobordism classes is a monomorphism in dimension four, two compact oriented spinor space-times are spinor-cobordant (as Spin(4) manifolds) iff they have the same topological signature. The following relation generalises spin-cobordism.

Definition. Two compact, closed Spin(n) manifolds $(X_1, t_{X_1}^{+,s_1})$ and $(X_2, t_{X_2}^{+,s_2})$ are called linearly spin-cobordant iff there exists an oriented $n + 1$ manifold Z with a Spin(n)-bundle $z^{+,s}$ such that $z^{+,s} | X_1 = t_{X_1}^{+,s_1}$ and $z^{+,s} | X_2 = t_{X_2}^{-,s_2}$ on $\partial_0 Z = X_1^+ \cup X_2^-$ where the complementary line bundle on Z is inner normal on X_1 and exterior normal on X_2.

Linear spin-cobordism is an equivalence relation and there is a group $M^{L\text{-Spin}}$ of linear spin-cobordism classes graded into subgroups of cobordism classes of n-dimensional spin manifolds $M_n^{L\text{-Spin}}$. In the following lemma we compute the group of cobordism classes of compact spinor space-times regarded as Spin(4) manifolds.

Lemma 6. Two compact spinor space-times are linearly spin-cobordant iff they are spin-cobordant.

Proof. Clearly two linearly spin-cobordant space-times must be spin-cobordant (if (Z, z), with z a Spin(4)-bundle is a linear spin-cobordism between two compact four-dimensional spin-manifolds, $w_2(z) = 0$ implies that $w_2(t_Z) = 0$ and therefore that Z is a Spin(5)-manifold in such a way that the Spin(5)-structure on Z induces a spin-cobordism.) Conversely, suppose that two compact, four-dimensional, spin-manifolds of Euler number zero are spin-cobordant.

Then there exists a compact five-dimensional spin-manifold Z such that $\partial_0 Z^+ = X_1^+ \cup X_2^-$ and a Spin(5)-structure on t_Z^+ such that the interior normal trivialisation of the normal bundle on X_1 induces the Spin(4)-structure on t_{X_1} and the exterior normal trivialisation of the normal bundle on X_2 induces the preassigned Spin(4)-structure on t_{X_2}. We next show that t_Z splits off a trivial line bundle interior normal on X_1 and exterior normal on X_2. To see this, suppose that $N = X_2 xI$ is a closed collar of X_2 in Z and suppose that $\overset{*}{Z}$ denotes the manifold $Z - (N - X_2 xO)$. Then $\overset{*}{Z}$ is diffeo to Z and therefore $\chi(\overset{*}{Z}) = \chi(Z) = 0$ because $\chi(2Z) = 2\chi(Z) - \chi(X_1) - \chi(X_2) = 0$ and $\chi(X_1) = \chi(X_2) = 0$. By the Poincaré-Hopf theorem (Milnor, 1965) $\overset{*}{Z}$ and N admit one-frames interior oriented on the boundaries (respectively $X_1^+ \cup X_2^-$ and $X_2^+ \cup X_2^-$). By reversing the frame on N and gluing N to $\overset{*}{Z}$ along X_2 one obtains a one-frame field on Z interior normal on X_1 and exterior normal on X_2. Suppose that z is the complementary oriented four-plane field on Z. Then z is tangent to ∂Z and its orientation restricts to that of $t_{X_1}^+$ on X_1 and that of $t_{X_2}^-$ on X_2. Moreover, because $w_2(t_Z) = w_2(z) + w_1(z).w_1(z^\perp) = w_2(z) = 0$, z admits a Spin(4)-structure on its orthogonal $SO(4)$ frame bundle which, because of the spin-cobordism, restricts on X_1 and X_2 to the preassigned spin structures.

3. *Bordism and Concordance of Lorentzian Structures*

The purpose of this section is primarily to report on some very recent work of Koschorke on tangent line bundles on compact manifolds which I interpret in terms of Lorentzian structures using Steenrod's theorem. Firstly, Koschorke defined a bordism relation between tangent line bundles which answers a question I posed (Whiston, 1974) on the cobordism of Lorentzian structures. We have to use the notion of bordism of maps (Conner & Floyd, 1967) defined as a generalised homology theory. Fix a smooth manifold X. Then two smooth maps $f_0 : Y_0 \to X$ and $f_1 : Y_1 \to X$ from (oriented) n-manifolds Y_0, Y_1 into the manifold X are called bordant iff there exists an (oriented) $n + 1$ manifold Z with (oriented) boundary $Y_0^+ \cup Y_1^-$ and a smooth map $F: Z \to X$ restricting to f_0, f_1 on the boundary ∂Z. The set of (oriented) bordism classes of maps from (oriented) manifolds into X forms a graded ring $\mathcal{N}(X)$ (respectively $\Omega(X)$) called the (oriented) bordism ring of X graded by the subgroups $\mathcal{N}_n(X)$ (respectively $\Omega_n(X)$) of (oriented) bordism classes of maps of n-manifolds into X.

Suppose that the pairs (X_0, L_0) and (X_1, L_1) consist of an oriented n-manifold and a tangent line bundle on the manifold. Then (X_0, L_0) and (X_1, L_1) are said to be bordant iff there exists an oriented $n + 1$ manifold Z with a tangent line bundle L such that $\partial_0 Z = X_0^+ \cup X_1^-$ and $L \mid X_1 = L$ and $L \mid X_2 = L_2$. This bordism is an equivalence relation and there exist groups $M_n(1)$ of bordism classes of tangent line bundles on oriented n-manifolds. Suppose that (X_0, L_0) and (X_1, L_1) are bordant tangent line bundles and that $f_0, f_1 : X_0, X_1 \to B\mathbb{Z}_2$ are their classifying maps. Then f_0 and f_1 are bordant as maps into $B\mathbb{Z}_2$, i.e. define the same element of $\Omega_n(B\mathbb{Z}_2)$. This defines a forgetful map $M_n(1) \to \Omega_n(B\mathbb{Z}_2)$. Note that two maps into $B\mathbb{Z}_2$, regarded as classifying maps of not necessarily tangent line bundles on the domain manifolds, are bordant iff the pull-back bundles extend to a line bundle on the manifold realising the bordism. Although the calculations (via a 'super' Poincaré–Hopf theorem and relative bordism groups) are on the whole straightforward, the result which interests us here is that there is a short-exact sequence: $M_4(1) \overset{f}{\rightarrowtail} \Omega_4(B\mathbb{Z}_2) \overset{\chi_2}{\twoheadrightarrow} \mathbb{Z}_2$ where χ_2 sends a bordism class (X, f) into $\chi(X)\mathrm{mod}(2)$ (χ_2 is well-defined because cobordant manifolds have the same Euler numbers mod(2)). By a rather complicated geometrical argument, one can prove that $\Omega_q(B\mathbb{Z}_2) \cong M_q^{SO} \oplus M_{q-1}^{0}$. Therefore because $M_3^{O} = 0$ (any two three-manifolds are cobordant in the unoriented sense), one can replace $\Omega_4(B\mathbb{Z}_2)$ in the above short-exact sequence by M_4^{SO}. This means that $M_4(1)$, the set of bordism classes of tangent line bundles on oriented four-manifolds, is isomorphic to $\ker(\chi_2)$ which (as I pointed out in Whiston, 1974) is the group of oriented cobordism classes of four-manifolds of Euler number zero. Therefore two tangent line bundles (X_0, L_0) and (X_1, L_1) are bordant iff the manifolds X_0 and X_1 are cobordant iff $\mathrm{sig}(X_0^+) = \mathrm{sig}(X_1^+)$. In particular, any two tangent line bundles on a given compact, orientable four-manifold are bordant and therefore any two oriented (and time-oriented) Lorentzian structures on a given space-time are cobordant.

Suppose that L_1 and L_2 are two tangent line bundles on a manifold X. Koschorke calls L_1 and L_2 'concordant' iff there is a tangent line bundle L on $X \times I$ with $L \mid X \times 0 = L_0$ and $L \mid X \times 1 = L_1$, and proves the following remarkable result: if X is an even dimensional, orientable and connected manifold of Euler number zero, there are $2^{a(X)+1} - 1$ concordance classes of tangent line bundles on X where $a(X) = \dim_{Z_2}(H^1(X, \mathbb{Z}_2))$. For a four-manifold of Euler number zero, $0 = \Sigma_{i=0}^{4} (-1)^i \dim_{Z_2}(H^i(X, \mathbb{Z}_2))$. Therefore if X is such that $H^i(X, \mathbb{Z}_2) \cong H_i(X, \mathbb{Z}_2) \pitchfork_{\mathbb{Z}_2} \mathbb{Z}_2$, Poincaré duality implies that $a(X) \geqslant 1$ and therefore that there are at least three concordance classes of Lorentzian structures on X.

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